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# Transformations Reducing the Order of the Parameter in Differential Eigenvalue Problems

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A family of order-reducing transformations applicable to a wide class of differential eigenvalue problems with nonlinear parameter dependence is developed. The highest or the first few highest powers of the parameter are removed, leading to the increased efficiency of the global numerical eigenvalue-search scheme of choice (taken to be spectral in this work). For unbounded-domain problems this cost reduction is accompanied by an increased accuracy and increased searching capability of the spectral technique. Applications to the spatial stability of the Orr–Sommerfeld problems for channel, boundary-layer, and wake flows are addressed explicitly. © 1988 Academic Press, Inc.

## 1. INTRODUCTION

In this paper a series of transformations of the dependent variable of differential eigenvalue problems with non-linear parameter dependence are presented. These transformations are applicable to a wide class of problems in which the range of the independent variable may be bounded or unbounded. The eigenvalues will be invariant under such transformations, as will be shown. The highest power (or the first few highest powers) of the eigenvalue will, however, be reduced in the new formulation. This result shall be referred to as a "reduction in order" of the eigenvalue throughout this work. Furthermore, the term "nonlinear" shall denote the nonlinearity in the parameter and not of the operator. These reductions do afford some analytic simplifications in cases where the solution may so be obtained. However, most problems of interest require a numerical search for locating their eigenvalues in the complex plane of the parameter. It is for this latter class that the ramifications of the technique are greatest. The technique of choice in this paper is one due to Bridges and Morris ([1, 2]). In bounded domains, it consists of representing the eigenfunction in terms of a truncated Chebyshev series and determining the eigenvalues from the linear algebraic problem governing the coefficients of the Chebyshev polynomials. Much detail has been provided in [1], with extensions to the unbounded case in [2] where use is made of an algebraic mapping of the $[0, \infty)$ interval onto $[-1, 1)$ prior to the Chebyshev series expansion.



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The goal of the present endeavor is not to develop a new eigenvalue search technique, but rather to advocate a preliminary analytical treatment of the equations which results in substantial cost reductions regardless of the particular global scheme used. For sake of definiteness and illustration, the technique will be applied to the problem of linear spatial stability of laminar flows. The governing equation there is the Orr–Sommerfeld equation wherein the eigenvalue appears up to and including the fourth power. It will be shown that the operation count can be reduced by a factor of eight. In the case of unbounded flows, this reduction in cost is accompanied by an increase in accuracy of the higher modes, for a given number of Chebyshev polynomials. The increased accuracy is a result of the increased area of the complex plane which is searched by the scheme, as a result of the transformation.

Through these transformations, *all* of the spatial eigenvalues may be obtained, whereas before, only a subset of the complex plane could be searched. Application of the search to the untransformed Orr–Sommerfeld equation in unbounded domain would yield the near neutral modes directly. The least stable mode could then be tracked from this mode [2]. For most practical purposes the above mentioned modes are all that are needed. However, when an arbitrary disturbance is to be expanded in terms of spatially evolving modes [3], then all the eigenvalues are needed, hence the necessity of the transformations.

To motivate the approach, an illustrative example is provided in Section 2. In Section 3 the technique is formally introduced in terms of general linear operators in bounded domains. The application to the spatial stability of channel flow is also included in Section 3. In Section 4 the spatial stability of unbounded shear flows is addressed and the technique is generalized to unbounded domains.

## 2. A Simple Example

Let us consider the following differential eigenvalue problem in the interval $y \in [-1, 1]$, reproduced from [1]:

$$D^2\phi - 2\alpha\omega\, D\phi + \alpha^2\phi = 0, \tag{1}$$

$$\phi(-1) = \phi(1) = 0, \tag{2}$$

where $D \equiv d/dy$ and $\omega$ is a prescribed parameter. For any given $\omega$, the eigenvalues $\alpha$ and their corresponding eigenfunctions, $\phi$, are sought. An exact solution exists for this simple case and is given in [1].

To find the values of $\alpha$ numerically, $\phi$ is expanded in a truncated Chebyshev series:

$$\phi(y) = \sum_{n=0}^{N}{}' \, a_n T_n(y), \tag{3}$$

where prime denotes that the first term of (3) has to be divided by two. $T_n(y)$ is the $n$th order Chebyshev polynomial of the first kind. The truncation of the series is equivalent to solving a perturbation of (1). This method is known as Lanczo's tau method. The perturbation, for (1) integrated twice, has the form

$$\tau_1 T_{N+1}(y) + \tau_2 T_{N+2}(y). \tag{4}$$

$\tau$'s are a measure of error in the solution. One can easily show their magnitude in this problem to be $O(\alpha/N)$, so that the resolution of the higher modes requires larger $N$.

Upon substitution of (3) into (1) and (2), the differential eigenvalue problem reduces to a nonlinear algebraic eigenvalue problem with the eigenvector $\mathbf{a}_\phi$ being the vector of the expansion coefficients, $a_n$, for $\phi$ as defined in (3):

$$\mathbf{D}_2(\alpha)\, \mathbf{a}_\phi = 0. \tag{5}$$

$\mathbf{D}_2$ is a lambda matrix [4] of degree two, which may be written as

$$[\mathbf{C}_2\alpha^2 + \mathbf{C}_1\alpha + \mathbf{C}_0]\, \mathbf{a}_\phi = 0, \tag{6}$$

where the $\mathbf{C}_i$ ($i = 1, 2, 3$) are independent of $\alpha$. The eigenvalues correspond to the values of $\alpha$ which render the determinant of $\mathbf{D}_2(\alpha)$ zero. System (6) may be transformed into one wherein the eigenvalue appears linearly, but the size of the system is doubled [1]. Using a $QR$ or $QZ$ algorithm to find the eigenvalues of this larger system requires an amount of work proportional to $O([2N]^3)$.

Now going back to (1) and (2), consider the following change of the independent variable:

$$\phi = \psi \exp(A\alpha y), \tag{7}$$

where $A$ is a constant to be determined. Substitution of (7) into (1) and (2) yields

$$[D^2 + 2\alpha(A-\omega)D + (A^2 - 2\omega A + 1)\,\alpha^2]\psi = 0 \tag{8}$$

$$\psi(-1) = \psi(1) = 0 \tag{9}$$

as the governing equation for $\psi$. The fact that the exponential function has no zeroes in the finite $\alpha$-plane ensures a nontrivial $\psi$ whenever $\phi$ is nontrivial. Therefore, the eigenvalues are invariant under the proposed transformation. $A$ is now chosen so that the coefficient of $\alpha^2$ in (8) is zero. There are two solutions for $A$:

$$A = \omega \pm (\omega^2 - 1)^{1/2} \equiv \omega + C(\omega). \tag{10}$$

Either choice is suitable. The resulting form of (8) is

$$[D^2 + 2C(\omega)\,\alpha D]\psi = 0. \tag{11}$$

The form of (11) is much simpler than (1). When $\psi$ is expanded in a finite Chebyshev series, (11) reduces to the linear algebraic eigenvalue problem

$$\mathbf{D}_1(\alpha)\,\mathbf{a}_\psi = 0, \tag{12}$$

where $D_1$ is now a lambda matrix of degree one. As a result there is no need to double the size of the system, $QR$ or $QZ$ methods being applicable in the present form. The operation count in this case is only proportional to $O(N^3)$. Therefore, in this simple example, using the transformation (7), the operation count was reduced by a factor of eight. If one checks $\tau_1$, it is still $O(\alpha/N)$, so that no amount of accuracy is compromised.

In the next section we shall show where and why such transformations are applicable.

## 3. Generalization on Finite Intervals

Consider the following general eigenvalue problem in operator form, with $y \in [a, b]$:

$$L_{y;\alpha}\phi(y;\alpha) = 0 \tag{13}$$

$$M^1_{y;\alpha\phi(a;\alpha)} = M^2_{y;\alpha}\phi(b;\alpha) = 0, \tag{14}$$

where

$$L \equiv \sum_{i=0}^{N} \sum_{j=0}^{N-i} a_{ij}(y)\,\alpha^j D^i. \tag{15}$$

$M^1$ and $M^2$ are similar in form to $L$ except that they are of one lower differential order. It is helpful to consider the following decomposition of $L$:

$$L \equiv \sum_{i=0}^{N} a_{i,N-i}\alpha^{N-i}D^i + \sum_{i=0}^{N-1} \sum_{j=0}^{N-i-1} a_{i,j}\alpha^j D^i \tag{16a}$$

$$\equiv L_b + L_u. \tag{16b}$$

$L_b$ is the "balanced" portion of $L$, namely the terms in which the sum of the power of the eigenvalue and the order of the derivative (to be referred to as the index) is equal to the order of the operator. $L_u$ is the unbalanced portion of $L$ which is formally $L - L_b$. In order for the proposed transformations to be applicable, $L$ has to satisfy several conditions:

(i) $\sum_{i=0}^{N-1} \sum_{j=i+1}^{N} |a_{N-i,i} \cdot a_{N-j,j}| \neq 0$ for all $y \in [a, b]$, so that there are always at least two nonzero terms.

(ii) $a_{0N} \neq 0$ for some $y \in [a, b]$, so that $\alpha^N$ is present in $L$.

(iii) coefficients of $L_b$ have no singularities for all $y \in [a, b]$.

Furthermore, we assume that $L$ is "irreducible" in $\alpha$, namely that the powers of $\alpha$ which appear in $L$ are prime with respect to each other. This last requirement is just to ensure that no simple redefinition of $\alpha$ would be able to trivially reduce the power of $\alpha$ in the operator. As an aside, let us mention that this definition of $L$, in general, may preclude the studying of problems which address non-isotropicity effects. There the index can be larger than the differential order for $L$ or $M_i$ ($i=1, 2$). The transformation to be considered is a generalization of (7):

$$\phi(y;\alpha)=\psi(y;\alpha)\exp\left[\alpha\int^{y}\zeta(y')\,dy'\right]. \tag{17}$$

Substitution of (17) into Eq. (13), and setting the coefficient of $\alpha^N$ equal to zero, yields an $N$th order polynomial equation for $\zeta(y)$ in the form

$$P_{N,y}[\zeta(y)]=0 \tag{18}$$

with $P_{N,y}(\cdot)\equiv\sum_{i=0}^{N}p_i(y)(\cdot)^i=\sum_{i=0}^{N}a_{i,N-1}(y)(\cdot)^i$. $P_N$ is a generalization of the characteristic polynomial for $L_b$, when the latter has non-constant coefficients. Conditions (i) and (iii) above ensure that there exists at least one root of $P_N$ which is an analytic function of $y$ on the whole interval $[a, b]$. In what follows, attention is restricted to the case $a_{N0}=1$ and $a_{0N}\neq 0$ on $[a, b]$, which is adequate for most problems of interest. As a result of this restriction, $p_N=1$ and there are $N$ possible solutions for (18), any one of which would be a sufficient choice. One possible approach to solving (18) is to locate a zero of $P_N$ with $y$ set equal to $a$, one of the end points. The following expression, which is obtained by differentiating (18) with respect to $y$, allows one to track this zero throughout the range of $y$:

$$\zeta'(y)=-\frac{\sum_{n=0}^{N}p_n'(y)\,\zeta^n(y)}{\sum_{n=1}^{N}np_n(y)\,\zeta^{n-1}(y)}. \tag{19}$$

The right-hand side of (19) will be well defined provided that the chosen zero is a simple zero of $P_N$ throughout the range of $y$. In other words the trajectory of this zero in the complex $\zeta$-plane as $y$ goes from $a$ to $b$, is not crossed by that of any other zeroes, at the same value of $y$.

The original problem, defined by (13) and (14) now reduces to

$$\tilde{L}_{y;\alpha}\psi(y;\alpha)=0 \tag{20}$$

$$\tilde{M}^1_{y;\alpha}\psi(a;\alpha)=0; \qquad \tilde{M}^2_{y;\alpha}\psi(b;\alpha)=0, \tag{21}$$

where $\tilde{L}$, $\tilde{M}^1$, and $\tilde{M}^2$ have the same form as their counterparts without the $\tilde{}$. In particular,

$$\tilde{L}\equiv\sum_{i=0}^{N}\sum_{j=0}^{N-i}a_{ij}(y)\,\alpha^jD^i. \tag{22}$$

The effect of the transformation though, has been to set the coefficient of $\alpha^N$ equal to zero:

$$\tilde{a}_{0N} \equiv 0. \tag{23}$$

Therefore, using a Chebyshev expansion scheme similar to that of Section 2, and solving for the eigenvalues of the discretized system, requires at most $[(N-1)/N]^3$ times work as that of the untransformed problem. The amount of savings would be greater if $L_b$ were factorizable. Before considering this case, one comment with regard to the nonfactorizable $L_b$'s is in order. As $N$ increases the amount of savings becomes less and the solution of $P_N$ could become difficult. For those cases the application of the technique might not be as desirable. However, for most eigenvalue problems arising from physical problems, $N$ is such that the technique should prove useful.

Before closing this section, let us discuss the conditions on $L$ for which the above transformation yields higher reductions in cost than $[(N-1)/N]^3$. If $L_b$ is factorizable into $k$ factors as

$$L_b = \prod_{i=1}^{k} L_i \tag{24}$$

then the implementation of the transformation becomes much simpler, since $L_i$'s are simpler than $L_b$ (recall the irreducibility of $L_b$). If it also happens that the chosen factor has a multiplicity higher than one, say $m$, then the reduction of the highest power of $\alpha$ in $L_b$ is by $m$. In such a case one should consider $L_u$, since the dominant power of $\alpha$ may now lie in $L_u$ such that the actual reduction of the power of $\alpha$ is less than $m$. In cases where the $\alpha$-content of $L_u$ is markedly subdominant to that of $L_b$, or that $L_u$ itself is reduced in $\alpha$ under the action of the proposed transformation, then the amount of savings in costs approaches $[(N-m)/N]^3$, which for a reasonable $N$ is quite substantial. One point has to be made with regard to the $L_i$'s, the factors of $L_b$. Since $L_b$ is a "balanced" operator, then so will be the $L_i$'s (with lower indices, of course). The imbalance of $L_i$'s would lead to an imbalance of $L$, which would be a contradiction. Use will be made of this observation in the discussion of problems with unbounded range of the independent variable. To illustrate the usefulness of the technique, the following example is provided.

### 3.1. *Linear Spatial Stability of Channel Flows*

Consider the spatial stability of small disturbances introduced in a channel flow, with $y \in [-1, 1]$, and $x \in (-\infty, \infty)$. The mean flow, $U(y)$, has the Poiseuille profile. The governing equation is the Orr–Sommerfeld equation

$$\{(D^2-\alpha^2)^2 - iR[(\alpha U - \omega)(D^2-\alpha^2) - \alpha U'']\}\phi = 0, \tag{25}$$

with

$$\phi = D\phi = 0 \qquad \text{on} \quad y = \pm 1, \tag{26}$$

where the streamfuncton of the small disturbances is given as

$$\phi(y;\alpha)\exp[i(\alpha x-\omega t)]. \tag{27}$$

$R$ is the Reynolds number based on the channel half-depth. $\omega$ is the real frequency, which is fixed for spatial studies. Prime indicates differentiation with respect to the argument. The instability or stability is governed by the sign of the imaginary part of $\alpha$. Further details may be found in texts on hydrodynamic stability (e.g., [5]). In their present form, (25) and (26) comprise a differential eigenvalue problem with $\alpha^4$ as the highest power of the eigenvalue, $\alpha$. If the Chebyshev polynomials are utilized again, and $NC$ of them are retained prior to the truncation, then the operation count to find the eigenvalues of the discretized system will be proportional to

$$O(4\times NC)^3. \tag{28}$$

The following transformation of the independent variable suggests itself:

$$\phi(y;\alpha)=\psi(y;\alpha)\exp(-\alpha y). \tag{29}$$

The formulation of the problem in terms of $\psi$ becomes

$$\{D^2(D-2\alpha)^2-iR[(\alpha U-\omega)(D-2\alpha)\,D-\alpha U'']\}\,\psi=0 \tag{30}$$

$$\psi(\pm 1)=D\psi(\pm 1)=0. \tag{31}$$

The highest power of $\alpha$ is reduced by two, because the balanced portion of the Orr–Sommerfeld operator is factorizable with multiplicity two, and the same factors appear in the dominant part of the unbalanced operator, with multiplicity one.

If the numerics are performed now, the amount of work will be proportional to

$$O(2\times NC)^3, \tag{32}$$

which in comparison with (28) represents a savings of 87.5 %, namely eight times faster, for the very same global numerical technique. The ramifications in the case of unbounded flows are even more spectacular, as will be demonstrated in the next section.

## 4. Extension to Unbounded Domains

Results of Section 3, as they stand, can be generalized to the cases where the range of the independent variable is $[0,\infty)$. However, what will be shown hereafter is that if $L$ satisfies some more specific conditions, then the reductions in cost will be accompanied by an immense increase in accurracy for higher modes. In order to develop the ideas, first an example, namely the Orr–Sommerfeld equation in the unbounded domain, is discussed. With this example behind us, the results are then generalized and some sufficiency conditions on $L$ are stated such that these transformations will attain their utmost usefulness.

### 4.1. *Linear Spatial Stability of Unbounded Shear Flows*

The streamfunction of small amplitude perturbations to the mean flow for flow over a rigid flat plate or in a wake, if the locally parallel-flow assumption is invoked, evolves according to the Orr–Sommerfeld equation, with a large Reynolds number based on some displacement length scale,

$$\{(D^2-\alpha^2)^2 - iR[(\alpha U-\omega)(D^2-\alpha^2)-\alpha U'']\}\,\phi = 0 \tag{33}$$

subject to

$$\phi,\ D\phi \to 0 \qquad \text{as} \quad y\to\infty, \tag{34}$$

and one of the following:

$$\phi(0) = D\phi(0) = 0, \tag{35a}$$

$$\phi(0) = D^2\phi(0) = 0, \tag{35b}$$

or

$$D\phi(0) = D^3\phi(0) = 0. \tag{35c}$$

Conditions (35a–35c) correspond to the no-slip condition for the flat plate, or odd or even conditions on the centerline for wake flow, respectively. Keller [6] has proposed a "correct" way of imposing the boundary conditions at infinity, should these conditions be imposed at a large but finite value of $y$. In the present case an algebraic mapping from $y \in [0, \infty)$ to $z \in [-1, 1)$ is employed which then allows for subsequent Chebyshev expansion:

$$z = (y - \tilde{1})/(y + \tilde{1}), \tag{36}$$

where $\tilde{1}$ is an $O(1)$ tuning parameter. As a result, the conditions (34) are truly imposed at infinity, and hence are correct as they stand. The numerical scheme to find the eigenvalues of (33) is clearly developed in [2], and it is shown that the conditions

$$\phi,\ D\phi \text{ bounded} \qquad \text{as } y\to\infty \tag{37}$$

are also covered by the scheme. This means that some mildly oscillatory continuous modes close to the the branch points are also recovered by the technique. However, as the continuous modes, in the spatial problem, become highly oscillatory upon moving along the branch cuts away from the branch points, "most" of these modes cannot be resolved. For details of the scheme the reader is referred to [2]. One point needs attention, however. Algebraic mapping of (36) has a metric $(m(z))$ which vanishes like a double zero as $z$ approaches 1. This, in general, would cause some ambiguity in this manner in which derivative boundary conditions at $z = 1$ should be prescribed, since

$$\frac{d\phi}{dy} = m(z)\,\frac{d\phi}{dz} \tag{38}$$

so that (34) is satisfied even when $d\phi/dz$ is nonzero. The resolution proposed in [2] is to define a new variable, $\xi$, as

$$\xi \equiv m(z)\, d\phi/dz. \tag{39}$$

The derivative boundary condition at $z=1$ can now simply be stated as $\xi = 0$.

A typical mode of the Orr–Sommerfeld problem has a tail which exhibits exponential decay and oscillatory behavior at infinity. In other words, the point at infinity ($y \to \infty$) is an essential singularity of the solution. When an algebraic mapping is performed on the function, unless the rate of decay is rather high, there will be an excessive amount of oscillations in the solution as $z \to 1$, due to the extreme contractions of the intervals approaching the point at infinity. Sufficient resolution of such high oscillations imposes a severe constraint on the minimum number of Chebyshev polynomials that are needed in our expansions. Another constraint on this number is obtained through insisting on the good resolution of the function near $z = -1$ where the function varies rapidly in the wall and critical layers. These constraints shall be referred to as the "first" and "second" constraints, respectively. The first constraint seems to be dominant, and a very non-sensitive function of $NC$, the number of polynomials retained.

Let us assume that an oscillatory function is resolved well, if there are at least 10 points per wavelength. The discretization points on the interval $z \in [-1, 1)$ correspond to the extrema of the $NC$th Chebyshev polynomial ($z_j \equiv \cos[\pi(1 + j/NC)]$; $j = 0, ..., NC$). The effect of the inverse mapping from $z$ to $y$ is to stretch the $\Delta z$ as the point $z = 1$ is approached. One can find the index $j^*$ of $z_j$ for which $\Delta y_j$ ($\equiv y_{j+1} - y_j$) is equal to one-tenth of the tail wavelength. Upon employing the definition (36) and some standard trigonometric identities, the index $j^*$ is given implicitly by

$$\tilde{l}\{\tan^2[\pi(j^*+1)/2NC] - \tan^2[\pi j^*/2NC]\} = \pi/(5\alpha_i). \tag{40}$$

Therefore, $j^* = j^*(\alpha_i)$. If $j^*$ is such that $|\phi_{j^*}| \equiv |\phi(y_{j^*})|$ is exponentially small, then the loss of resolution will not pose a major problem. In practice however, it seems that this norm has to be truly small or otherwise there will be resolution difficulties. The magnitude of the function is given (for large $y$) by

$$|\phi| = \exp(-\alpha_r y) \tag{41}$$

so that $|\phi(y_{j^*})| = \exp(-\alpha_r y_{j^*})$. By insisting on the smallness of $|\phi_{j^*}|$ ($\equiv \varepsilon$), and recalling that $j^* = j^*(\alpha_i)$, an expression is obtained relating $\alpha_r$ and $\alpha_i$: $\alpha_r = \alpha_r(\alpha_i)$. This is the equation of a line in the complex $\alpha$-plane. In fact there are two such lines which are reflections about the $\alpha_r$ axis. These lines pass through the origin, since for $\alpha_r = 0$, only modes with $\alpha_i = 0$ can be resolved. The wedge-like region bounded by these two lines is what will be referred to as the "validity region": if an eigenvalue resides inside this region, then the proposed numerical scheme will be able to detect it. As an aside let us mention that one of the branch cuts in the $\alpha$ plane, corresponding to the downstream-propagating viscous continuous modes, penetrates this region of validity. On this branch cut as well as a strip of width

$$\delta \simeq |(2\alpha_r)/(2\alpha_i + R)| \tag{42}$$

centered on this cut, the tail behavior of the eigenfunction is not $\exp(-\alpha y)$ but $\exp(-\mu y)$, where $\mu = [\alpha^2 + iR(\alpha - \omega)]^{1/2}$ with positive real part. The numerical scheme cannot resolve any discrete eigenvalues which fall on this "invalidity strip," say $S1$. In Fig. 1, a sketch of the "validity region" and its dependence on parameters $NC$ and $\varepsilon$ (the smallness of the tail amplitude) is provided. Curve (d) is the locus of the downstream-propagating viscous continuous modes for an arbitrary choice of $R = 500$ and $\omega = 0.4$. A region of width $\delta$ (42) centered on this line constitutes the "invalidity strip." Curve (b) bounds the validity region to its right, for the choice $NC = 30$ and $\varepsilon = 10^{-6}$. If at $\varepsilon = 10^{-6}$, $NC$ is increased to 60, curve (a) is obtained; and if at $NC = 30$, $\varepsilon$ is reduced to $10^{-8}$, curve (c) is obtained. Curves (a)–(c) have symmetric reflections about the $\alpha_r$-axis.

The validity region is a symmetrically disposed wedge about the real axis. Its wedge angle increases upon increasing $NC$. However, the rate is quite slow. The reason for the existence of this limited "region of validity" is the existence of an essential singularity at the point at infinity of $y$. Below we shall show that this singularity is removable, if an alternative formulation is adopted.

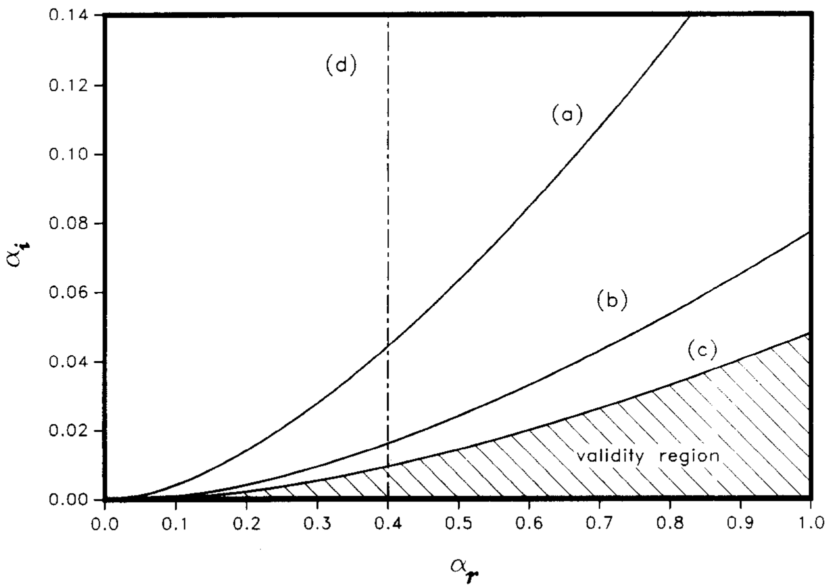


FIG. 1. Curves (a)–(c) denote the boundary of the region of validity lying to the right of the respective curves: (a) $NC = 60$, $\varepsilon = 10^{-6}$; (b) $NC = 30$, $\varepsilon = 10^{-6}$; (c) $NC = 30$, $\varepsilon = 10^{-8}$. (d) is the locus of the viscous continuous modes for $R = 500$ and $\omega = 0.4$. $\alpha_r$-axis is the locus of inviscid continuous modes. Curves (a)–(c) have reflections about the $\alpha_r$-axis.

In all of the complex $\alpha$-plane, except for $S1$ and its upstream propagating counterpart and the two inviscid branch cuts, the eigenfunction has the following asymptotic behavior:

$$\phi \simeq \exp(-\alpha y). \tag{43}$$

A new function $\psi$ is defined as

$$\psi = \phi \exp(\alpha y) \tag{44}$$

so that $\psi \to 1$ at infinity. The conditions to be imposed on $\psi$ at infinity are $D\psi = D^2\psi = 0$. The appearance of $D^2\psi$ necessitates the introduction of a new counterpart of $\xi$ (39) to resolve the ambiguities which arise because of the zero of the metric. $\psi$ has no singularities at infinity, and the "first" constraint is removed. Therefore, the validity region is now the whole of the complex $\alpha$-plane minus the points mentioned at the beginning of the paragraph (say set $A$). However, there still exists the second constraint to be reckoned with. But this is a mild constraint and with a reasonable number of polynomials, the region of validity approaches the whole of set $A$.

Therefore for a given number of polynomials it is now possible to recover many more eigenvalues and increase the accuracy of the ones that were resolved using the original formulation of the Orr–Sommerfeld equation. However, this is not all that the transformation (44) enables us to accomplish. There is a far greater bonus involved. The chosen transformation also happens to be the one that reduces the Orr–Sommerfeld operator in $\alpha$ by a factor of two. The governing equation for $\psi$ is

$$\{D^4 - 4\alpha D^3 + 4\alpha^2 D^2 - iR[(\alpha U - \omega)(D^2 - 2\alpha D) - \alpha U'']\}\,\psi = 0 \tag{45}$$

with boundary conditions which also involve a maximum power of $\alpha$ equal to two. The global scheme of [2] to find the eigenvalues of this system now only requires $\frac{1}{8}$ the work of that of the $\phi$-system.

To summarize the above results, the transformation (44) enabled us to resolve far more eigenvalues, more accurately, at a reduced cost ($\frac{1}{8}$ as before). In the following subsection we present some sufficiency criteria which, if satisfied by an operator, allow for such dramatic results.

### 4.2. *A Set of Sufficiency Criteria*

In this section we set forth a set of criteria which if satisfied by an operator, cause the accompaniment of the removal of the essential singularity at infinity, by a reduction in the computational costs. The conditions are not meant to be the most general possible; however, they do appear to encompass the problems of physical interest. They are

(i) The coefficients of $L$ are nonsingular on $y \in [0, \infty)$, and attain constant values at infinity.

(ii) The coefficient $a_{N0} \neq 0$ so that it may be set equal to 1 without loss of generality; and $a_{0N}$ does not vanish at infinity.

(iii) $L_b$ is factorizable into $L'L^*$, where $L^*$ is a first-order differential operator (with multiplicity $m1$). $L^*$ involves $\alpha$ explicitly because of (ii)). If $L_u$ is present, then it, too, should have $L^*$ as a factor at least at infinity.

(iv) The operator is stiff, so that there is a single dominant asymptotic behavior of the eigenfunction at infinity, governed by $L^*$. If $L_u$ is present then the following singular form of the operator, $L$, would ensure stiffness over most of the $\alpha$-plane

$$L \equiv L_b + RL_u \qquad (R \gg 1). \tag{46}$$

The asymptotic behavior of the eigenfunction will be given by the asymptotic form of $L^*$, in all of $\alpha$-plane, except for regions of width $(1/R)^n$ for some $n > 0$.

If $L$ satisfies these conditions, then the function

$$\psi = \phi \exp\left[\alpha \int^{y} \zeta(y')\, dy'\right] \tag{47}$$

with $\zeta$ chosen so as to reduce $L^*$ throughout $y \in [0, \infty)$, satisfies an equation which is lower in powers of $\alpha$ than that for $\phi$. The linear dependence of the argument of the exponential function in (47) on $\alpha$ is a consequence of $L^*$ being a balanced first-order operator, which explicitly involves $\alpha$. The power of $\alpha$ is reduced by at least $m3 = \min(m1, \Delta + m2)$, with $m2$ denoting the reduction of power of $\alpha$ in $L_u$, and $\Delta$ the original difference in the $\alpha$-content of $L_b$ and $L_u$. The asymptotic behavior of $\psi$ is now simple, namely,

$$\psi_{\text{asym}} = \phi_{\text{asym}} \exp\left[\alpha \int^{y \to \infty} \zeta(y')\, dy'\right] = \text{constant}. \tag{48}$$

This can be seen by recalling that $L^*$ is a first-order balanced differential operator, and the coefficient of $\alpha$ in $L^*$ approaches a nonzero constant at infinity.

Therefore, for such an operator, the transformation (47) will result in increased scope of the numerical search scheme, increased accuracy of the eigenvalues, and decreased costs of computation by $[(N - m3)/N]^3$. If (iii) is violated so that $L_b$ is only factorizable at infinity, then the reduction in cost may not be as appreciable.

## 5. Conclusions

It was shown that a family of simple transformations may be applied to nonlinear differential eigenvalue problems such that the order of the parameter is reduced. This reduction, for problems in bounded domains, leads to appreciable amount of savings in the number of operations necessary to determine "all" the eigenvalues.

For problems in unbounded domains the ramifications of such transformations are far greater. Grosch and Orszag [7] mention that for functions with simple behavior at infinity, the mapping of the interval results in increased accuracy. However, they count this method as not extremely applicable to problems where the functions exhibit oscillatory behavior at infinity (due to an essential singularity at infinity). In Section 4 it was shown that renormalizing the function by a multiple of its asymptotic behavior, for a class of operators, which also includes the Orr–Sommerfeld operator, leads to the removal of the singularity at infinity. Also the same savings as in the bounded domain case are realized. Therefore, it is possible to truly recover all the discrete eigenvalues with higher accuracy (for a given number of Chebyshev polynomials), at a reduced cost.

The application of such transformations (renormalizations) should not be restricted to eigenvalue problems. They could also be implemented in boundary value problems in infinite domains, where the solution exhibits a single dominant oscillatory behavior at infinity. The above results should rekindle a new interest in the use of mappings introduced in [7], at least for some class of operators.

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## REFERENCES

1. T. J. BRIDGES AND P. J. MORRIS, *J. Comput. Phys.* **55**, 437 (1984).
2. T. J. BRIDGES AND P. J. MORRIS, "Spectral Calculations of the Spatial Stability of Non-Parallel Boundary Layers" *AIAA 22nd Aerospace Sciences Meeting, Reno, Nevada, 1984, AIAA-84-0437.*
3. H. HAJ-HARIRI, Doctoral dissertation, MIT, 1987 (unpublished).
4. P. LANCASTER, *Lambda Matrices and Vibrating Systems* (Pergamon, Oxford, 1966).
5. C. C. LIN, *Theory of Hydrodynamic Stability* (Cambridge University Press, London, 1955).
6. H. B. KELLER, *Numerical Solution of Two Point Boundary Value Problems* (SIAM, Philadelphia, 1978).
7. C. E. GROSCH AND S. A. ORSZAG, *J. Comput. Phys.* **25**, 273 (1977).